An Extended Yellow Change Interval Solution Derived from GHM's Critical Distance

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Introduction

The *critical distance* is the foundation to Gazis, Herman and Maradudin's (GHM) universally adopted *minimum* yellow traffic signal timing solution presented in their 1960 paper, "The Problem of the Amber Signal Light in Traffic Flow" [1]. However, GHM's (permissive [2]) *minimum* yellow signal duration is limited in its application since they only accounted for vehicles moving at constant velocity traversing a *minimum* safe and comfortable stopping distance referenced a level intersection's stopping line, aka GHM's *critical distance*.

This abbreviated article presents a natural derivation to extend GHM's constant velocity solution to also include vehicle deceleration within the *critical distance*. It is achieved through the identification of the internal *boundary condition* defined by GHM's *minimum* braking distance, and by introducing a new intermediate/entry velocity variable. The resulting extended *minimum* yellow timing model is applicable to any approach which includes turning lanes where deceleration within the *critical distance* is required to perform safe and comfortable turns.

A key point to this article and GHM's paper, is that the *minimum* required yellow signal duration for a specific approach lane is directly defined by the *maximum unimpeded safe and comfortable spatial motion* of a vehicle and its passengers approaching, entering, and traveling through an intersection. Hence, the understanding of basic vehicle dynamics is imperative to properly implement traffic signal timings which is also why GHM's optional STOP or GO equations describe uniform (constant or average) vehicle motions, aka *kinematics.*

GHM's Model and Solution

The source to GHM's *minimum* GO (time) solution is their *minimum* safe and comfortable *distance* to STOP (GHM's *critical distance*) which is illustrated in Figure 1 with its Equation (1):

$$x_C = v_0 \cdot t_{PR} + \frac{v_0^2}{2a_{max}} \tag{1}$$

Where:

- x_C = Critical distance - the *minimum* safe and comfortable stopping distance, (ft or m)
- v_0 = *Maximum* uniform initial/approach velocity, (ft/s or m/s)
- t_{PR} = *Maximum* allocated driver-vehicle perception-reaction time, (s)
- a_{max} = *Maximum* uniform safe and comfortable deceleration, (ft/s^2 or m/s^2)

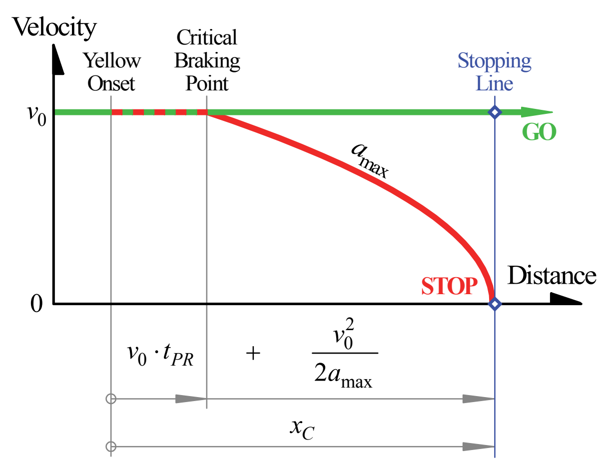

Figure 1 – Velocity vs distance profile for GHM's *minimum* STOP (the *critical distance*) or GO solutions

The first term of Equation (1) calculates the traveled distance at the *maximum* uniform initial/approach velocity (v_0) during the *maximum* allocated driver-vehicle perception-reaction time (t_{PR}) up until the critical braking point. This distance starts from the onset of the yellow signal indication until the driver-vehicle complex perceives and reacts to either STOP (foot on brake pedal) or GO (no action - foot on accelerator pedal maintaining constant velocity). It is during this *maximum* allowed distance and time drivers must decide whether they should STOP or GO.

The second term of Equation (1) calculates the *minimum* braking distance starting from the critical braking point at the *maximum* uniform initial/approach velocity (v_0) and ending at the stopping line with zero end velocity. The *minimum* braking distance is directly defined by GHM's *maximum* uniform safe and comfortable deceleration value (a_{max}). Hence, the *maximum* uniform safe and comfortable deceleration trajectory (a_{max}) ending at the stopping line is the actual *boundary condition* between being able to STOP or GO and the true source to GHM's solution as they also defined in their paper's very first equation.

GHM's Distance to Time Conversion

GHM simply divided their *critical distance* (x_C) Equation (1) with the *maximum* uniform (constant) initial/approach velocity (v_0) to convert the *minimum* STOP distance to the *minimum* GO time or the *minimum* permissive yellow signal duration (Y_P):

$$Y_P \geq \frac{x_C}{v_0} = \frac{v_0 \cdot t_{PR} + \frac{v_0^2}{2a_{max}}}{v_0} \tag{2}$$

Simplification of Equation (2) yields GHM's *minimum* permissive yellow signal timing equation ($v_0 > 0$):

$$\boxed{Y_P \geq t_{PR} + \frac{v_0}{2a_{max}}} \tag{3}$$

Equation (3) gives GHM's *minimum* yellow interval solution at constant vehicle velocity and it is limited to traffic flow in one spatial dimension or in other words, straight line travel through a level intersection. It *does not account* for safe and comfortable turning maneuvers which require vehicle deceleration and motion in an additional spatial dimension.

GHM also addressed in their original paper that their model and solution did not account for vehicles making turns but that their scientific methods could be applied to analyze turning maneuvers when they wrote:

> *"There are other variations to the problem of the dilemma zone such as the case of a vehicle approaching an intersection at slow speed with the intention of making a turn. This is a case of known practical difficulty and some information can be obtained from the present analysis with w taken equal to the distance traversed while turning."*

Given the fact that the original GHM model for signal timing is lacking to include vehicle deceleration prior to entering an intersection, a new model is needed. Since the goal is to derive an accurate expression for signal timing, a comprehensive understanding of vehicular motion dynamics through the intersection is imperative because time and distance are inseparably related. However, the distance traversed is constant which is also what GHM presented in their original paper when they defined the source to their solution - the *critical distance*.

Extending GHM's Model and Solution

An extension to GHM's original model can be developed by understanding its true function, which is *drivers faced with the onset of the yellow indication at the critical distance must make a decision to either STOP or GO before the critical braking point and follow through with their decision.* However, as already described, GHM's *minimum* permissive yellow duration only allows the driver-vehicle complex to travel the remaining braking distance at constant (or accelerated) velocity, their *minimum* GO solution.

Following GHM's logic, their model can be extended to also include deceleration within the *critical distance* by introducing a new intermediate/entry velocity variable (v_1) equal to or less than the initial/approach velocity (v_0). This new variable (v_1) allows for vehicle deceleration across the critical distance following the *boundary condition* set forth by the defined *maximum* uniform safe and comfortable deceleration trajectory (a_{max}) ending at the intersection's stopping line. The velocity vs distance STOP motion profile in Figure 1 is thus divided into three sections (rather than two) and the new intermediate/entry velocity variable (v_1) is introduced as shown in Figure 2 with its Equation (4):

$$x_C = v_0 \cdot t_{PR} + \frac{v_0^2 - v_1^2}{2a_{max}} + \frac{v_1^2}{2a_{max}} \tag{4}$$

Where ($v_0 \geq v_1 \geq 0$):

x_C = Critical distance - the *minimum* safe and comfortable stopping distance, (ft or m)
t_{PR} = *Maximum* allocated driver-vehicle perception-reaction time, (s)
v_0 = *Maximum* uniform initial/approach velocity, (ft/s or m/s)
v_1 = *Maximum* uniform intermediate/entry velocity, (ft/s or m/s)
a_{max} = *Maximum* uniform safe and comfortable deceleration, (ft/s^2 or m/s^2)

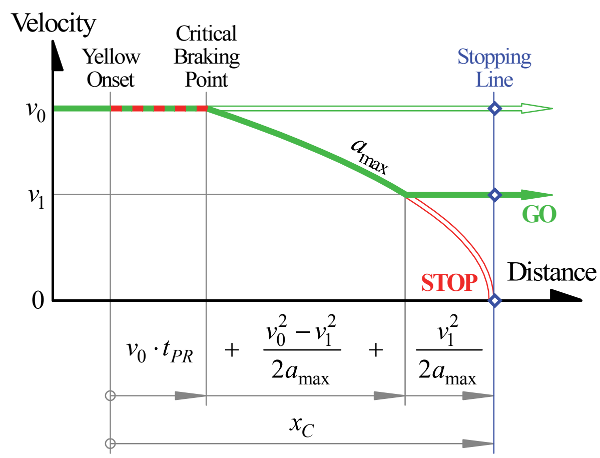

Figure 2 – Velocity vs distance for GHM's critical distance including an intermediate/entry velocity

The second term of Equation (4) calculates the traveled distance from the critical braking point decelerating at the *maximum* uniform safe and comfortable deceleration (a_{max}) (the STOP or GO boundary) from the *maximum* uniform initial/approach velocity (v_0) to the *maximum* uniform intermediate/entry velocity (v_1). The third term calculates the remaining distance to the stopping line so that GHM's original *critical distance* (x_C) is maintained.

The Extended Model and Solution's Distance to Time Conversion

Factorization of Equation (4)'s second *distance* term using the conjugate rule [3] reveals the term's *velocity* and *time* products:

$$x_C = v_0 \cdot t_{PR} + \overbrace{\frac{(v_0 + v_1)}{2}}^{\text{Velocity}} \cdot \overbrace{\frac{(v_0 - v_1)}{a_{max}}}^{\text{Time}} + \frac{v_1^2}{2a_{max}} \tag{5}$$

Next, as GHM, divide Equation (5)'s three distance terms using the correct initial/approach, average and intermediate/entry vehicle velocities to convert the *critical distance* (x_C) to the *extended minimum* permissive yellow signal duration (Y_{EP}):

$$Y_{EP} \geq \frac{v_0 \cdot t_{PR}}{v_0} + \frac{\frac{(v_0 + v_1)}{2} \cdot \frac{(v_0 - v_1)}{a_{max}}}{\frac{(v_0 + v_1)}{2}} + \frac{\frac{v_1^2}{2a_{max}}}{v_1} \tag{6}$$

Reduction of Equation (6) gives the basic *extended minimum* GO time solution derived from GHM's *critical distance*:

$$Y_{EP} \geq t_{PR} + \frac{(v_0 - v_1)}{a_{max}} + \frac{v_1}{2a_{max}} \tag{7}$$

Further simplification using a common denominator yields:

$$\boxed{Y_{EP} \geq t_{PR} + \frac{2v_0 - v_1}{2a_{max}}} \tag{8}$$

Or in a different form:

$$\boxed{Y_{EP} \geq t_{PR} + \frac{v_0}{a_{max}} - \frac{v_1}{2a_{max}}} \tag{9}$$

Where $v_0 \geq v_1 > 0$

The above extended *minimum* permissive yellow signal duration or GO solution seen in Equations (8) and (9) will yield GHM's original Equation (3) if $v_1 = v_0$ (constant velocity). Furthermore, setting $v_1 = 0$ (zero end velocity), the equations will yield the *minimum* time it takes to STOP traversing the *critical distance*. Hence, the yellow signal duration should always be less than the *minimum* time it takes to STOP as Equation (9) shows with its negative third term. If not, a driver in a vehicle might end up still faced with a yellow indication at the stopping line. In addition, as GHM pointed out, the yellow signal *duration* is irrelevant for a vehicle that does not enter the intersection which is also the reason why the extended GO solution's entry velocity variable (v_1) is defined to be greater than zero.

Summary

The purpose of this brief article is to show the function and limitation of GHM's original solution and to present a natural extension expanding upon GHM's logic resulting in a universal solution applicable to any intersections' approach including turning lanes. It is the vehicle's motion and path through the intersection that ultimately determines the time necessary to traverse the intersection which becomes the principal source to its change interval.

GHM's solution is limited to constant velocity through an intersection. Whereas the presented extended solution also includes vehicle deceleration through the introduction of a new entry velocity variable. Thus providing a minimum solution applicable for any maneuver or type of vehicle, including autonomous vehicles, traversing a level intersection.

The scope of this article is limited to level intersections which was the case of GHM's original paper as well. The effects of Earth's gravity due to road grade or varying friction coefficients due to road and weather conditions, as well as, how to properly implement system tolerances including variations in human behaviors will be addressed in a later article or paper. In addition, using state-of-the-art vehicle dynamics test instrumentation, expressions to calculate the new entry velocity variable for the driver-vehicle complex traveling through an intersection's geometry will be presented.

References and Notations

1. D. Gazis, R. Herman and A. A. Maradudin, "The Problem of the Amber Signal Light in Traffic Flow", *Operations Research,* vol. 8, no. 1, pp. 112-132, 1960. http://jarlstrom.com/PDF/The_Problem_Of_The_Amber_Signal_Light_In_Traffic_Flow.pdf

2. A *permissive* yellow signal refers to when the yellow signal indication's *only function* is to *warn* traffic of the impending signal change to red and the right-of-way.
3. Example of a *conjugate rule*: $(a+b)(a-b) = a^2-b^2$